Constraint propagation & backtracking-based search

Roman Barták
Charles University, Prague (CZ)

roman.bartak@mff.cuni.cz
http://ktiml.mff.cuni.cz/~bartak

„Constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it.“

Eugene C. Freuder, Constraints, April 1997

Holly Grail of Programming

```
> Computer, solve the SEND, MORE, MONEY problem!

> Here you are. The solution is
  [9,5,6,7]+[1,0,8,5]=[1,0,6,5,2]
```

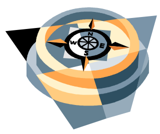
a Star Trek view

```
> Sol=[S,E,N,D,M,O,R,Y],
  domain([E,N,D,O,R,Y],0,9), domain([S,M],1,9),
           1000*S + 100*E + 10*N + D +
           1000*M + 100*O + 10*R + E #=
  10000*M + 1000*O + 100*N + 10*E + Y,
  all_different(Sol),
  labeling([ff],Sol).

> Sol = [9,5,6,7,1,0,8,2]
```

today reality

Tutorial outline

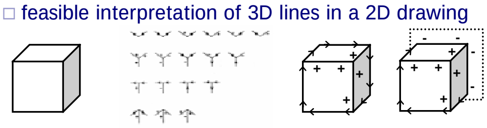
- **Introduction**
  - history, applications, a constraint satisfaction problem
- **Depth-first search**
  - backtracking, backjumping, backmarking
  - incomplete and discrepancy search
- **Consistency**
  - node, arc, and path consistencies
  - k-consistency and global constraints
- **Combining search and consistency**
  - look-back and look-ahead schemes
  - variable and value ordering
- **Conclusions**
  - constraint solvers, resources



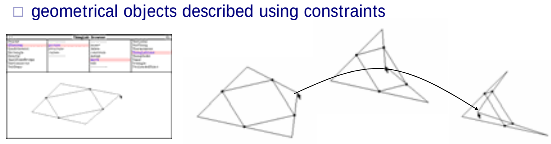
Introduction

A bit of history

- **Scene labelling** (Waltz 1975)
  - feasible interpretation of 3D lines in a 2D drawing



- **Interactive graphics** (Sutherland 1963, Borning 1981)
  - geometrical objects described using constraints

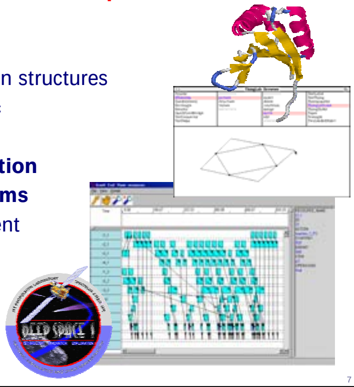


- **Logic programming** (Gallaire 1985, Jaffar, Lassez 1987)
  - from unification to constraint satisfaction

Application areas

All types of hard combinatorial problems:

- **molecular biology**
  - DNA sequencing
  - determining protein structures
- **interactive graphic**
  - web layout
- **network configuration**
- **assignment problems**
  - personal assignment
  - stand allocation
- **timetabling**
- **scheduling**
- **planning**



7

Constraint technology

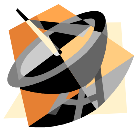
based on **declarative problem description** via:

- **variables with domains** (sets of possible values)
  e.g. start of activity with time windows
- **constraints** restricting combinations of variables
  e.g. endA < startB

constraint **optimization** via objective function
e.g. minimize makespan

Why to use constraint technology?

- understandable
- open and extendible
- proof of concept



8

CSP

Constraint satisfaction problem consists of:

- a finite set of **variables**
- **domains** - a finite set of values for each variable
- a finite set of **constraints**
  - constraint is an arbitrary relation over the set of variables
  - can be defined extensionally (a set of compatible tuples) or intentionally (formula)

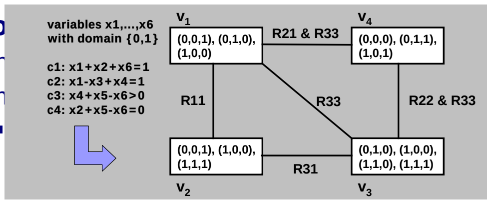
A **solution** to a CSP is a complete consistent assignment of variables.

- **complete** = a value is assigned to every variable
- **consistent** = all the constraints are satisfied

9

Two or more?

- **Binary constraint satisfaction**
  - only binary constraints
  - any CSP is convertible to a binary CSP
    - **dual encoding (Stergiou & Walsh, 1990)**
      swapping the role of variables and constraints
- **Boo**
  - on
  - an



10

Two or more?

- **Binary constraint satisfaction**
  - only binary constraints
  - any CSP is convertible to a binary CSP
    - **dual encoding (Stergiou & Walsh, 1990)**
      swapping the role of variables and constraints
- **Boolean constraint satisfaction**
  - only Boolean (two valued) domains
  - any CSP is convertible to a Boolean CSP
    - **SAT encoding**
      Boolean variable indicates whether a given value is assigned to the variable

11

Depth-first search

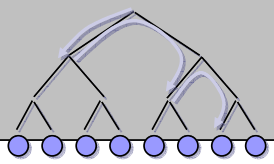
Basic idea

- We are looking for a **complete consistent assignment**!
  - start with a **consistent assignment** (for example, empty one)
  - **extend the assignment** towards a complete assignment
- **Depth-first search** is a technique of searching solution by extending a partial consistent assignment towards a complete consistent assignment.
  - **assign values** gradually to variables
  - after each assignment **test consistency** of the constraints over the assigned variables
  - and **backtrack** upon failure
- **Backtracking** is probably the most widely used complete systematic search algorithm.
  - complete = guarantees finding a solution or proving its non-existence

Chronological backtracking

A recursive definition

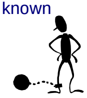
```
procedure BT(X:variables, V:assignment, C:constraints)
    if X={} then return V
    x ← select a not-yet assigned variable from X
    for each value h from the domain of x do
        if consistent(V+{x/h}, C) then
            R ← BT(X-{x}, V+{x/h}, C)
            if R≠fail then return R
    end for
    return fail
end BT

call BT(X, {}, C)
```



Consistency procedure checks satisfaction of constraints whose variables are already assigned.

Weaknesses of backtracking

- **thrashing**
  - throws away the reason of the conflict
  - **Example:** A,B,C,D,E:: 1..10, A>E
    - BT tries all the assignments for B,C,D before finding that A≠1
  - **Solution:** backjumping (jump to the source of the failure)
- **redundant work**
  - unnecessary constraint checks are repeated
  - **Example:** A,B,C,D,E:: 1..10, B+8<D, C=5*E
    - when labelling C,E the values 1,..,9 are repeatedly checked for D
  - **Solution:** backmarking, backchecking (remember (no-)good assignments)
- **late detection of the conflict**
  - constraint violation is discovered only when the values are known
  - **Example:** A,B,C,D,E::1..10, A=3*E
    - the fact that A>2 is discovered when labelling E
  - **Solution:** forward checking (forward check of constraints)



Backjumping

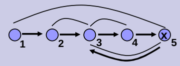
Backjumping is a technique for **removing thrashing** from backtracking.

How?

1) identify the source of the conflict (impossibility to assign a value)
2) jump to the past variable in conflict

- The same forward run like in backtracking, only the back-jump can be longer, and hence irrelevant assignments are skipped!
- **How to find a jump position? What is the source of the conflict?**
  - select the constraints containing just the currently assigned variable and the past variables
  - select the closest variable participating in the selected constraints

Graph-directed backjumping



Enhancement: use only the violated constraints

✎ **conflict-directed backjumping**

Conflict-directed backjumping in practice

N-queens problem

	A	B	C	D	E	F	G	H
1	♛							
2			♛					
3					♛			
4		♛						
5				♛				
6	1	3,4	2,5	4,5	3,5	1	2	3
7								
8								

Queens in rows are allocated to columns.

6th queen cannot be allocated!

1. Write a number of conflicting queens to each position.

2. Select the farthest conflicting queen for each position.

3. Select the closest conflicting queen among positions.

Note:

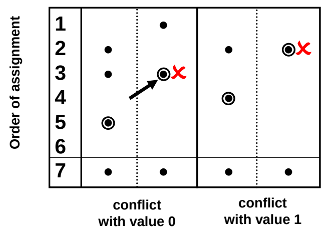
Graph-directed backjumping has no effect here (due to the complete graph)!

Conflicting variable

How to find out the conflicting variable?

Situation:

- assume that the variable no. 7 is being assigned (values are 0, 1)
- the symbol • marks the variables participating the violated constraints (two constraints for each value)



Neither 0 nor 1 can be assigned to the seventh variable!

1. Find the closest variable in each violated constraint (o).

2. Select the farthest variable from the above chosen variables for each value (✗).

3. Choose the closest variable from the conflicting variables selected for each value and jump to it.

Consistency check (BJ)

In addition to the test of satisfaction of the constraints, the closest conflicting level is computed

```
procedure consistent(Labeled, Constraints, Level)
    J ← Level           % the level to which we will jump
    NoConflict ← true   % indicator of a conflict
    for each C in Constraints do
      if all variables from C are Labeled then
         if C is not satisfied by Labeled then
            NoConflict ← false
            J ← min {J, max{L | X in C & X/V/L in Labeled & L<Level}}
         end if
      end if
    end for
    if NoConflict then return true
                  else return fail(J)
end consistent
```


Gaschnig (1979)

Algorithm backjumping

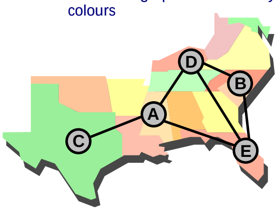
```
procedure BJ(Unlabeled, Labeled, Constraints, PreviousLevel)
    if Unlabelled = {} then return Labeled
    pick first X from Unlabelled
    Level ← PreviousLevel+1
    Jump ← 0
    for each value V from DX do
        C ← consistent({X/V/Level} ∪ Labeled, Constraints, Level)
        if C = fail(J) then
            Jump ← max {Jump, J}
        else
            Jump ← PreviousLevel
            R ← BJ(Unlabeled-{X},{X/V/Level} ∪ Labeled,Constraints, Level)
            if R ≠ fail(Level) then return R   % success or backjump
        end if
    end for
    return fail(Jump)        % jump to the conflicting variable
end BJ

call BJ(Variables,{},Constraints,0)
```


Weakness of backjumping

When jumping back the in-between assignment is lost!

Example:

- colour the graph in such a way that the connected vertices have different colours



node	vertex	
A	1	1
B	2	1
C	*1* 2	*1* 2
D	*1 2* 3	*1* 2
E	*1 2 3*	*1 2* 3

backjump

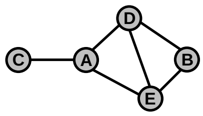
During the second attempt to label C superfluous work is done
- it is enough to leave there the original value 2, the change of B does not influence C.

Ginsberg (1993)

Dynamic backtracking

example

The same graph (A,B,C,D,E), the same colours (1,2,3) but a different approach.



Backjumping
+ remember the source of the conflict
+ carry the source of the conflict
+ change the order of variables

= DYNAMIC BACKTRACKING

node	1	2	3
A	•		
B		•	
C	A	•	
D	A	B	•
E	A	B	D

jump back
+ carry the conflict source

node	1	2	3
A	•		
B		•	
C	A	•	
D	A	B	AB
E	A	B	

jump back
+ carry the conflict source
+ change the order of B, C

node	1	2	3
A	•		
C	A	•	
B	•	A	
D	A	•	
E	A	B	•

• selected colour
AB a source of the conflict

The vertex C (and the possible sub-graph connected to C) is not re-coloured.

Ginsberg (1993)

Algorithm dynamic BT

```
procedure DB(Variables, Constraints)
    Labelled ← {};   Unlabelled ← Variables
    while Unlabelled ≠ {} do
        select X in Unlabelled
        ValuesX ← DX - {values inconsistent with Labelled using Constraints}
        if ValuesX = {}  then
            let E be an explanation of the conflict (set of conflicting variables)
            if E = {}  then failure
            else
                let Y be the most recent variable in E
                unassign Y (from Labelled) with eliminating explanation E-{Y}
                remove all the explanations involving Y
            end if
        else
            select V in ValuesX
            Unlabelled ← Unlabelled - {X}
            Labelled ← Labelled ∪ {X/V}
        end if
    end while
    return Labelled
end DB
```

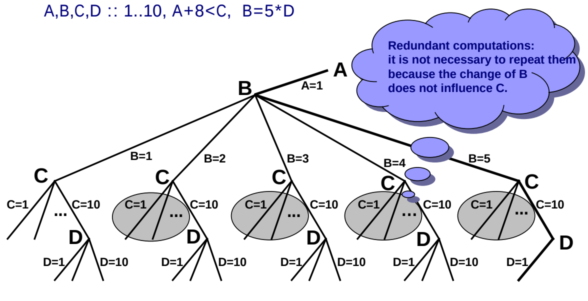

Redundant work

What is redundant work in chronological backtracking?

- repeated computation whose result has already been obtained

Example:

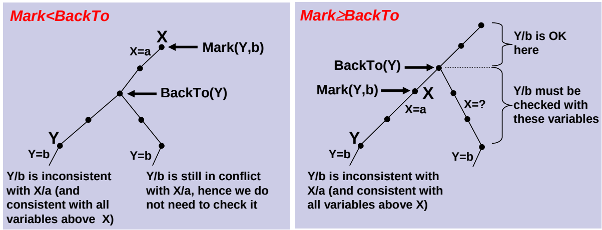
A,B,C,D :: 1..10, A+8<C, B=5*D



Haralick, Elliot (1980)

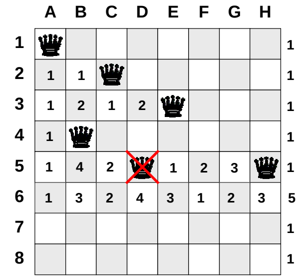
Backmarking

- Removes redundant constraint checks by memorizing negative and positive tests:
  - **Mark(X,V)** is the farthest (instantiated) variable in conflict with the assignment X=V
  - **BackTo(X)** is the farthest variable to which we backtracked since the last attempt to instantiate X
- Now, some constraint checks can be omitted:



Backmarking in practice

N-queens problem



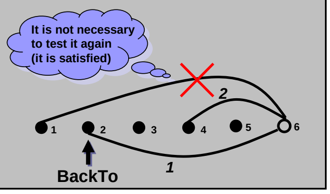
1. Queens in rows are allocated to columns.
2. Latest choice level is written next to chessboard (BackTo). At beginning 1s.
3. Farthest conflict queen at each position (MarkTo). At beginning 1s.
4. 6th queen cannot be allocated!
5. Backtrack to 5, change BackTo.
6. When allocating 6th queen, all the positions are still wrong (MarkTo<BackTo).

Note:
Backmarking can be combined with backjumping (for free).

Consistency check (BM)

Only the constraints where any value is changed are re-checked, and the farthest conflicting level is computed.

```
procedure consistent(X/V, Labeled, Constraints, Level)
    for each Y/VY/LY in Labeled such that LY≥BackTo(X) do
        % only possible changed variables Y are explored
        % in the increasing order of LY (first the oldest one)
        if X/V is not compatible with Y/VY using Constraints then
            Mark(X,V) ← LY
            return fail
        end if
    end for
    Mark(X,V) ← Level-1
    return true
end consistent
```



Algorithm backmarking

```
procedure BM(Unlabeled, Labeled, Constraints, Level)
  if Unlabelled = {} then return Labeled
  pick first X from Unlabelled          % fix order of variables
  for each value V from D_X do
    if Mark(X,V) ≥ BackTo(X) then % re-check the value
      if consistent(X/V, Labeled, Constraints, Level) then
        R ← BM(Unlabeled-{X}, Labeled ∪{X/V/Level}, Constraints, Level+1)
        if R ≠ fail then return R       % solution found
      end if
    end if
  end for
  BackTo(X) ← Level-1                   % jump will be to the previous variable
  for each Y in Unlabelled do           % tell everyone about the jump
    BackTo(Y) ← min {Level-1, BackTo(Y)}
  end for
  return fail                           % return to the previous variable
end BM
```

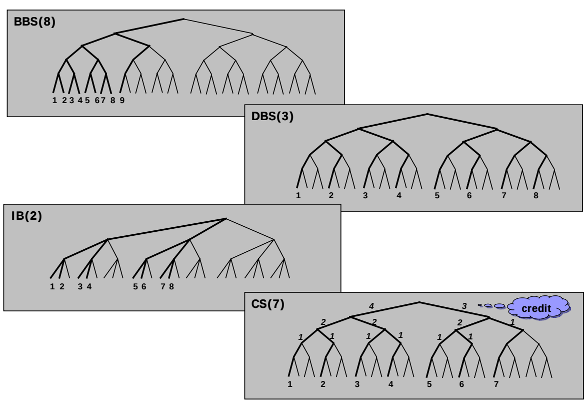

Incomplete search

A **cutoff limit** to stop exploring a (sub-)tree
- some branches are skipped → incomplete search

When no solution found, **restart** with enlarged cutoff limit.

- **Bounded Backtrack Search (Harvey, 1995)**
  - restricted number of backtracks
- **Depth-bounded Backtrack Search (Cheadle et al., 2003)**
  - restricted depth where alternatives are explored
- **Iterative Broadening (Ginsberg and Harvey, 1990)**
  - restricted breadth in each node
  - still exponential!
- **Credit Search (Beldiceanu et al., 1997)**
  - limited credit for exploring alternatives
  - credit is split among the alternatives

Incomplete search



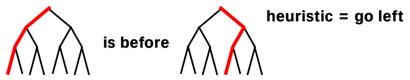
Heuristics in search

- **Observation 1:**
  The **search space** for real-life problems is so **huge** that it cannot be fully explored.
- **Heuristics** - a guide of search
  - they recommend a value for assignment
  - quite often lead to a solution
- What to do upon a **failure of the heuristic**?
  - BT cares about the end of search (a bottom part of the search tree)
    so it rather repairs later assignments than the earliest ones
    thus BT assumes that the heuristic guides it well in the top part
- **Observation 2:**
  The **heuristics** are **less reliable in the earlier parts** of the search tree (as search proceeds, more information is available).
- **Observation 3:**
  The number of **heuristic violations** is usually **small**.

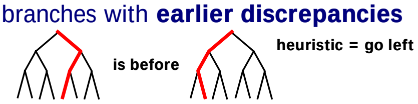
Discrepancies

Discrepancy
= the heuristic is not followed

Basic principles of discrepancy search:
change the order of branches to be explored
- prefer branches with **less discrepancies**

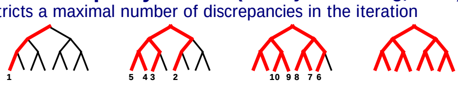


- prefer branches with **earlier discrepancies**

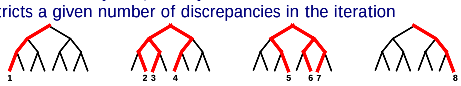


Discrepancy search

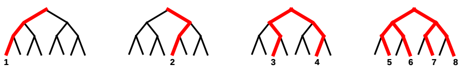
- **Limited Discrepancy Search (Harvey & Ginsberg, 1995)**
  - restricts a maximal number of discrepancies in the iteration



- **Improved LDS (Korf, 1996)**
  - restricts a given number of discrepancies in the iteration



- **Depth-bounded Discrepancy Search (Walsh, 1997)**
  - restricts discrepancies till a given depth in the iteration



- **…**

*** heuristic = go left**

Consistency

Introduction to consistencies

So far we used constraints in a **passive way** (as a test) …
…in the best case we analysed the reason of the conflict.

Cannot we use the constraints in a more active way?

Example:

A in 3..7, B in 1..5 — the variables' domains
A<B — the constraint

many inconsistent values can be removed
we get A in 3..4, B in 4..5

Note: it does not mean that all the remaining combinations of the values are consistent (for example A=4, B=4 is not consistent)

How to remove the inconsistent values from the variables' domains in the constraint network?

Node consistency (NC)

Unary constraints are converted into variables' domains.

Definition:
- **The vertex** representing the variable X is **node consistent** iff every value in the variable's domain D_x satisfies all the unary constraints imposed on the variable X.
- **CSP** is **node consistent** iff all the vertices are node consistent.

Algorithm NC

```
procedure NC(G)
    for each variable X in nodes(G) do
        for each value V in the domain D_X do
            if unary constraint on X is inconsistent with V then
                delete V from D_X
        end for
    end for
end NC
```

Arc consistency (AC)

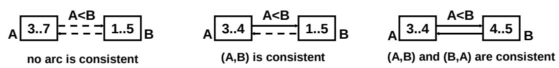
Since now we will assume binary CSP only

i.e. a constraint corresponds to an arc (edge) in the constraint network.

Definition:

- **The arc (V_i,V_j)** is **arc consistent** iff for each value x from the domain D_i there exists a value y in the domain D_j such that the assignment V_i =x and V_j = y satisfies all the binary constraintson V_i, V_j.
  - **Note:** The concept of arc consistency is directional, i.e., arc consistency of (V_i,V_j) does not guarantee consistency of (V_j,V_i).
- **CSP** is **arc consistent** iff every arc (V_i,V_j) is arc consistent (in both directions).

Example:



Arc revisions

How to make (V_i,V_j) arc consistent?

- Delete all the values x from the domain D_i that are inconsistent with all the values in D_j (there is no value y in D_j such that the assignment V_i = x, V_j = y satisfies all the binary constrains on V_i a V_j).

Algorithm of arc revision

```
procedure REVISE((i,j))
    DELETED ← false
    for each X in Di do
        if there is no such Y in Dj such that (X,Y) is consistent, i.e.,
              (X,Y) satisfies all the constraints on Vi, Vj then
            delete X from Di
            DELETED ← true
        end if
    end for
    return DELETED
end REVISE
```

The procedure also reports the deletion of some value.

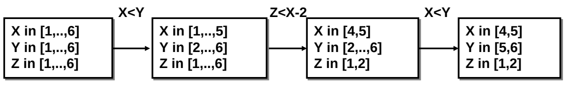
Mackworth (1977)

Algorithm AC-1

How to establish arc consistency among the constraints?

Doing revision of every arc is not enough!

Example: X in [1,..,6], Y in [1,..,6], Z in [1,..,6], X<Y, Z<X-2



Make all the constraints consistent until any domain is changed.

Algorithm AC-1

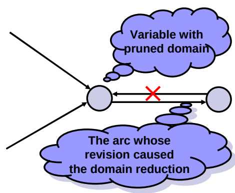
```
procedure AC-1(G)
    repeat
        CHANGED ← false
        for each arc (i,j) in G do
            CHANGED ← REVISE((i,j)) or CHANGED
        end for
    until not(CHANGED)
end AC-1
```

What is wrong with AC-1?

If a single domain is pruned then revisions of all the arcs are repeated even if the pruned domain does not influence most of these arcs.

What arcs should be reconsidered for revisions?

- The arcs whose consistency is affected by the domain pruning, i.e., the **arcs pointing to the changed variable**.
- **We can omit one more arc!**
  - Omit the arc running out of the variable whose domain has been changed (this arc is not affected by the domain change).



Mackworth (1977)

Algorithm AC-2

A generalised version of the **Waltz's labelling algorithm**.

In every step, the arcs going back from a given vertex are processed (i.e. a sub-graph of visited nodes is AC)

Algorithm AC-2

```
procedure AC-2(G)
    for i ← 1 to n do                          % n is a number of variables
        Q ← {(i,j) | (i,j)∈arcs(G), j<i}        % arcs for the base revision
        Q' ← {(j,i) | (i,j)∈arcs(G), j<i}       % arcs for re-revision
        while Q non empty do
            while Q non empty do
                select and delete (k,m) from Q
                if REVISE((k,m)) then
                    Q' ← Q' ∪ {(p,k) | (p,k)∈arcs(G), p≤i, p≠m }
            end while
            Q ← Q'
            Q' ← empty
        end while
    end for
end AC-2
```

Mackworth (1977)

Algorithm AC-3

Re-revisions can be done **more elegantly than in AC-2**.

1) **one queue** of arcs for (re-)revisions is enough

2) only the **arcs affected by domain reduction** are added to the queue (like AC-2)

Algorithm AC-3

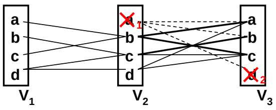
```
procedure AC-3(G)
    Q ← {(i,j) | (i,j)∈arcs(G), i≠j}    % queue of arcs for revision
    while Q non empty do
        select and delete (k,m) from Q
        if REVISE((k,m)) then
            Q ← Q ∪ {(i,k) | (i,k)∈arcs(G), i≠k, i≠m}
        end if
    end while
end AC-3
```

AC-3 schema is the most widely used consistency algorithm but it is still not optimal (time complexity is $O(ed^3)$).

Looking for a support

Observation (AC-3):

- Many pairs of values are tested for consistency in every arc revision.
- These tests are repeated every time the arc is revised.



1. When the arc V_2,V_1 is revised, the value a is removed from domain of V_2.
2. Now the domain of V_3, should be explored to find out if any value a,b,c,d loses the support in V_2.

Observation:

The values a,b,c need not be checked again because they still have a support in V_2 different from a.

The support set for a∈D_i is the set {<j,b> | b∈D_j , (a,b)∈$C_{i,j}$}

Cannot we compute the support sets once and then use them during re-revisions?

Computing support sets

A set of values supported by a given value (if the value disappears then these values lost one support), and a number of own supports are kept.

```
procedure INITIALIZE(G)
   Q ← {} , S ← {}                          % emptying the data structures
   for each arc (Vi,Vj) in arcs(G) do
      for each a in Di do
         total ← 0
         for each b in Dj do
            if (a,b) is consistent according to the constraint Ci,j then
               total ← total + 1
               Sj,b ← Sj,b ∪ {<i,a>}
            end if
         end for
         counter[(i,j),a] ← total
         if counter[(i,j),a] = 0 then
            delete a from Di
            Q ← Q ∪ {<i,a>}
         end if
      end for
   end for
   return Q
end INITIALIZE
```

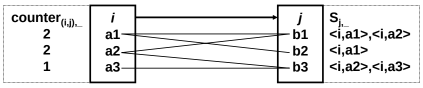
***$S_{j,b}$* - a set of pairs <i,a> such that <j,b> supports them**

***counter[(i,j),a]* - number of supports for the value *a* from D_i in the variable V_j**

Using support sets

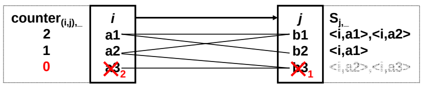
Situation:

we have just processed the arc (i,j) in INITIALIAZE



Using the support sets:

1. Let b3 is deleted from the domain of j (for some reason).
2. Look at $S_{j,b3}$ to find out the values that were supported by b3 (i.e. <i,a2>,<i,a3>).
3. Decrease the counter for these values (i.e. tell them that they lost one support).
4. If any counter becomes zero (a3) then delete the value and repeat the procedure with the respective value (i.e., go to 1).



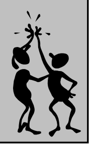
Algorithm AC-4

Mohr, Henderson (1986)

The algorithm AC-4 has **optimal worst case time complexity O(ed²)**!

Algorithm AC-4

```
procedure AC-4(G)
   Q ← INITIALIZE(G)
   while Q non empty do
      select and delete any pair <j,b> from Q
      for each <i,a> from Sj,b do
         counter[(i,j),a] ← counter[(i,j),a] - 1
         if counter[(i,j),a] = 0 & "a" is still in Di then
              delete "a" from Di
              Q ← Q ∪ {<i,a>}
         end if
      end for
   end while
end AC-4
```



Unfortunately the **average time complexity is not so good** ... plus there is a **big memory consumption**!

Other AC algorithms

- **AC-5 (Van Hentenryck, Deville, Teng, 1992)**
  - generic AC algorithm covering both AC-4 and AC-3
- **AC-6 (Bessière, 1994)**
  - improves AC-4 by remembering just one support
- **AC-7 (Bessière, Freuder, Régin, 1999)**
  - improves AC-6 by exploiting symmetry of the constraint
- **AC-2000 (Bessière & Régin, 2001)**
  - an adaptive version of AC-3 that either looks for a support or propagates deletions
- **AC-2001 (Bessière & Régin, 2001)**
  - improvement of AC-3 to get optimality (queue of variables)
- **AC-3.1 (Zhang & Yap, 2001)**
  - improvement of AC-3 to get optimality (queue of constraints)

…

Directional arc consistency (DAC)

Observation 1:

AC has a directional character but a CSP is not directional.

Observation 2:

AC has to repeat arc revisions; the total number of revisions depends on the number of arcs but also on the size of domains (while cycle).

Is it possible to weaken AC in such a way that every arc is revised just once?

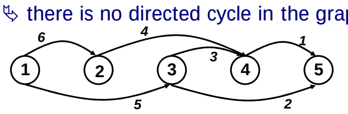
Definition:

CSP is **directional arc consistent** using a given order of variables iff every arc (i,j) such that i<j is arc consistent.

Again, every arc has to be revised, but revision in one direction is enough now.

Algorithm DAC-1

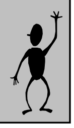
1) Consistency of an arc is required just in one direction.
2) Variables are ordered
   - there is no directed cycle in the graph!



If arcs are explored in a „good" order, no revision has to be repeated!

Algorithm DAC-1

```
procedure DAC-1(G)
    for j = |nodes(G)| to 1 by -1 do
        for each arc (i,j) in G such that i<j do
            REVISE((i,j))
        end for
    end for
end DAC-1
```



49

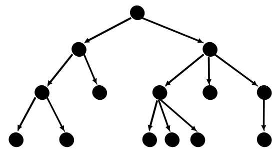
How to use DAC?

AC visibly covers DAC (if CSP is AC then it is DAC as well)

So, is DAC useful?
- DAC-1 is surely much faster than any AC-x
- there exist problems where DAC is enough

Claim:

If the constraint graph forms a tree then DAC is enough to solve the problem without backtracks.
- How to order the vertices for DAC?
- How to order the vertices for search?



1. Apply DAC in the order from the root to the leaf nodes.

2. Label vertices starting from the root.

DAC guarantees that there is a value for the child node compatible with all the parents.

50

Relation between DAC and AC

Observation:

CSP is arc consistent iff for some order of the variables, the problem is directional arc consistent in both directions.

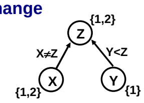
Is it possible to achieve AC by applying DAC in both primal and reverse direction?

In general NO, but ...

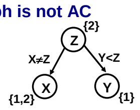
Example:

X in {1,2}, Y in {1}, Z in {1,2}, X≠Z, Y<Z

using the order X,Y,Z there is no domain change



using the order Z,Y,X, the domain of Z is changed but the graph is not AC



However if the order Z,Y,X is used first then we get AC!

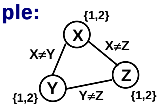
51

Is arc consistency enough?

By using AC we can remove many incompatible values
- **Do we get a solution?**
- **Do we know that there exists a solution?**

Unfortunately, the answer to both above questions is NO!

Example:



CSP is arc consistent but there is no solution

So what is the benefit of AC?

Sometimes we have a solution after AC
- any domain is empty → no solution exists
- all the domains are singleton → we have a solution

In general, AC prunes the search space.

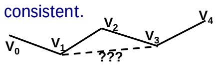
52

Path consistency (PC)

How to strengthen the consistency level?

More constraints are assumed **together**!

Definition:
- **The path** $(V_0,V_1,\ldots, V_m)$ is **path consistent** iff for every pair of values $x \in D_0$ a $y \in D_m$ satisfying all the binary constraints on V_0,V_m there exists an assignment of variables $V_1,\ldots,V_{m-1}$ such that all the binary constraints between the neighbouring variables V_i,V_{i+1} are satisfied.
- CSP is **path consistent** iff every path is consistent.



Some notes:
- only the **constraints between the neighboring variables** must be satisfied
- it is enough to explore **paths of length 2** (Montanary, 1974)

53

Relation between PC and AC

Does PC subsume AC (i.e. if CSP is PC, is it AC as well)?
- the arc (i, j) is consistent (AC) if the path **(i,j,i)** is consistent (PC)
- thus PC implies AC

Is PC stronger than AC (is there any CSP that is AC but it is not PC)?

Example:

X in {1,2}, Y in {1,2}, Z in {1,2}, X≠Z, X≠Y, Y≠Z
- it is AC, but not PC (X=1, Z=2 cannot be extended to X,Y,Z)

AC removes incompatible values from the domains, what will be done in PC?
- PC removes pairs of values
- PC makes constraints explicit (A<B,B<C ⇒ A+1<C)
- a unary constraint = a variable's domain

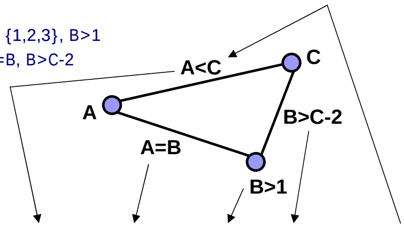
54

Path revision

Constraints **represented extensionally** via matrixes.
Path consistency is realized via **matrix operations**

Example:

- A,B,C in {1,2,3}, B>1
- A<C, A=B, B>C-2



$$\begin{pmatrix}011\\001\\000\end{pmatrix} \& \begin{pmatrix}100\\010\\001\end{pmatrix} * \begin{pmatrix}000\\010\\001\end{pmatrix} * \begin{pmatrix}110\\111\\111\end{pmatrix} = \begin{pmatrix}000\\001\\000\end{pmatrix}$$

55

Mackworth (1977)

Algorithm PC-1

How to make the path (i,k,j) consistent?

$R_{ij} \leftarrow R_{ij} \ \& \ (R_{ik} * R_{kk} * R_{kj})$

How to make a CSP path consistent?

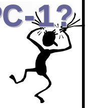
Repeated revisions of paths (of length 2) while any domain changes.

procedure PC-1(Vars,Constraints)
  n ← |Vars|, Y^n ← Constraints
  repeat
    $Y^0 \leftarrow Y^n$
    for k = 1 to n **do**
      for i = 1 to n **do**
        for j = 1 to n **do**
          $Y^k_{ij} \leftarrow Y^{k-1}_{ij} \ \& \ (Y^{k-1}_{ik} * Y^{k-1}_{kk} * Y^{k-1}_{kj})$
  until $Y^n=Y^0$
  Constraints ← Y^0
end PC-1

If we use $Y^k_{ii} \leftarrow Y^{k-1}_{ii} \ \& \ (Y^{k-1}_{ik} * Y^{k-1}_{kk} * Y^{k-1}_{ki})$ then we get AC-1

56

How to improve PC-1?



Is there any inefficiency in PC-1?

- just a few „bits“
  - it is not necessary to keep all copies of Y^k one copy and a bit indicating the change is enough
  - some operations produce no modification ($Y^k_{kk} = Y^{k-1}_{kk}$)
  - half of the operations can be removed ($Y_{ji} = Y^T_{ij}$)
- **the grand problem**
  - after domain change all the paths are re-revised but it is enough to revise just the influenced paths

Algorithm of path revision

procedure REVISE_PATH((i,k,j))
  Z ← $Y_{ij} \ \& \ (Y_{ik} * Y_{kk} * Y_{kj})$
  if Z=Y_{ij} **then return** false
  Y_{ij} ← Z
  return true
end REVISE_PATH

If the domain is pruned then the influenced paths will be revised.

57

Influenced paths

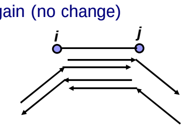
Because $Y_{ji} = Y^t_{ij}$ it is enough to revise only the paths (i,k,j) where i≤j.
Let the domain of the constraint (i,j) be changed when revising (i,k,j):

Situation a: i<j

all the paths containing (i,j) or (j,i) must be re-revised
but the paths (i,j,j), (i,i,j) are not revised again (no change)

S_a = {(i,j,m) | i ≤ m ≤ n & m≠j}
∪ {(m,i,j) | 1 ≤ m ≤ j & m≠i}
∪ {(j,i,m) | j < m ≤ n}
∪ {(m,j,i) | 1 ≤ m < i}

| S_a | = 2n-2



Situation b: i=j

all the paths containing i in the middle of the path are re-revised
but the paths (i,i,i) and (k,i,k) are not revised again

S_b = {(p,i,m) | 1 ≤ m ≤ n & 1 ≤ p ≤ m} - {(i,i,i),(k,i,k)}

| S_b | = n*(n-1)/2 - 2

58

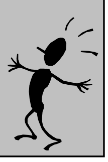
Mackworth (1977)

Algorithm PC-2

Paths in one direction only (attention, this is not DPC!)
After every revision, the **affected paths are re-revised**

Algorithm PC-2

procedure PC-2(G)
  n ← |nodes(G)|
  Q ← {(i,k,j) | 1 ≤ i ≤ j ≤ n & i≠k & j≠k}
  while Q non empty **do**
    select and delete (i,k,j) from Q
    if REVISE_PATH((i,k,j)) **then**
      Q ← Q ∪ RELATED_PATHS((i,k,j))
  end while
end PC-2



procedure RELATED_PATHS((i,k,j))
  if i<j **then return** S_a **else return** S_b
end RELATED_PATHS

59

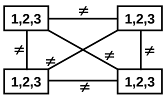
Other PC algorithms

- **PC-3 (Mohr, Henderson 1986)**
  - based on computing supports for a value (like AC-4)
    - If the pair (a,b) at the arc (i,j) is not supported by another variable, then a is removed from D_i and b is removed from D_j.
  - **this algorithm is not sound!**
- **PC-4 (Han, Lee 1988)**
  - correction of the PC-3 algorithm
  - based on computing supports of pairs (b,c) at arc (i,j)
- **PC-5 (Singh 1995)**
  - uses the ideas behind AC-6
  - only one support is kept and a new support is looked for when the current support is lost

60

Drawbacks of PC

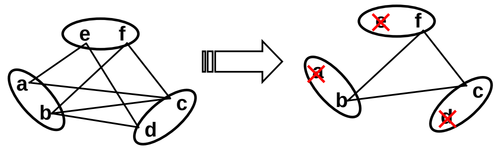
- **memory consumption**
  - because PC eliminates pairs of values, we need to keep all the compatible pairs extensionally, e.g. using {0,1}-matrix
- **bad ratio strength/efficiency**
  - PC removes more (or same) inconsistencies than AC, but the strength/efficiency ratio is much worse than for AC
- **modifies the constraint network**
  - PC adds redundant arcs (constraints) and thus it changes connectivity of the constraint network
  - this complicates using heuristics derived from the structure of the constraint network (like density, graph width etc.)
- **PC is still not a complete technique**
  - A,B,C,D in {1,2,3}
    A≠B, A≠C, A≠D, B≠C, B≠D, C≠D
    is PC but has no solution



Half way between AC and PC

Can we make a consistency algorithm:
- **stronger than AC**,
- **without drawbacks of PC** (memory consumption, changing the constraint network)?

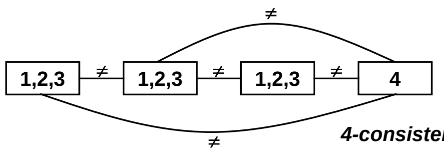
Restricted path consistency (Berlandier 1995)
- based on AC-4 (uses the support sets)
- as soon as a value has only one support in another variable, PC is evoked for this pair of values



k-consistency

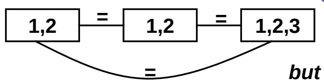
Is there a common formalism for AC and PC?
- AC: a value is extended to another variable
- PC: a pair of values is extended to another variable
- ... we can continue

Definition:
CSP is k-consistent iff any consistent assignment of (k-1) different variables can be extended to a consistent assignment of one additional variable.



4-consistent graph

Strong k-consistency



3-consistent graph
but not 2-consistent graph!

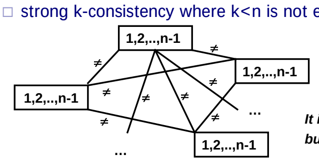
Definition:
CSP is strongly k-consistent iff it is j-consistent for every j≤k.

Visibly: **strong k-consistency ⇒ k-consistency**
Moreover: **strong k-consistency ⇒ j-consistency ∀j≤k**
In general: **¬ k-consistency ⇒ strong k-consistency**

- NC = strong 1-consistency = 1-consistency
- AC = (strong) 2-consistency
- PC = (strong) 3-consistency
  - sometimes we call NC+AC+PC together **strong path consistency**

What k-consistency is enough?

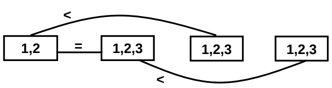
- Assume that the number of vertices is n. What level of consistency do we need to find out the solution?
- **Strong n-consistency for graphs with n vertices!**
  - n-consistency is not enough - see the previous example
  - strong k-consistency where k<n is not enough as well



graph with n vertices
domains 1..(n-1)

It is strongly k-consistent for k<n
but it has no solution

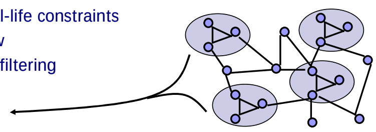
And what about this graph?



(D)AC is enough!
Because this a tree..

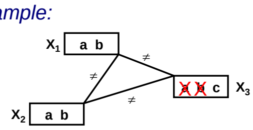
Think globally

CSP describes the problem locally:
the constraints restrict small sets of variables
- \+ heterogeneous real-life constraints
- \- missing global view
  - weaker domain filtering



Global constraints
- global reasoning over a local sub-problem
- using semantic information to improve efficiency

Example:

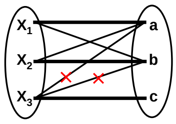


- local (arc) consistency deduces no pruning
- but some values can be removed

Régin (1994)

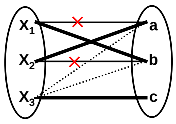
Inside all-different

- a set of binary inequality constraints among all variables
  $X_1 \neq X_2, X_1 \neq X_3, \ldots, X_{k-1} \neq X_k$
- **all_different({X_1,…, X_k})** = {(d_1,…, d_k) | $\forall i\ d_i \in D_i\ \&\ \forall i \neq j\ d_i \neq d_j$}
- better pruning based on **matching theory over bipartite graphs**



Initialisation:

1) compute **maximum matching**
2) **remove** all **edges** that do not belong to any maximum matching



Propagation of deletions ($X_1 \neq a$):

1) **remove** discharged **edges**
2) compute **new maximum matching**
3) **remove** all **edges** that do not belong to any maximum matching

Search and Propagation

How to solve CSPs?

So far we have two separate methods:

- **depth-first search**
  - complete (finds a solution or proves its non-existence)
  - too slow (exponential)
    - explores "visibly" wrong valuations
- **consistency techniques**
  - usually incomplete (inconsistent values stay in domains)
  - pretty fast (polynomial)

Share advantages of both approaches - combine them!

- label the variables step by step (backtracking)
- maintain consistency after assigning a value

Do not forget about traditional solving techniques!

- Linear equality solvers, simplex …
- such techniques can be integrated to global constraints!

There is also local search.

Core search procedure - DFS

The basic constraint satisfaction technology:

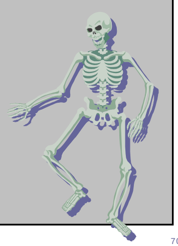
- **label the variables step by step**
  - the variables are marked by numbers and labelled in a given order
- **ensure consistency** after variable assignment

A skeleton of search procedure

procedure Labelling(G)
  return LBL(G,1)
end Labelling

procedure LBL(G,cv)
  if cv>|nodes(G)| **then return** nodes(G)
  for each value V from D_{cv} **do**
    if *consistent*(G,cv) **then**
      R ← LBL(G,cv+1)
      if R ≠ fail **then** return R
    end if
  end for
  return fail
end LBL

A „hook" for consistency procedure



Look back techniques

"Maintain" consistency among the already labelled variables.

„look back" = look to already labelled variables

What's **result of consistency maintenance** among labelled variables?

a conflict (and/or its source - a violated constraint)

Backtracking is the basic look back method.

Backward consistency checks

procedure AC-BT(G,cv)
  Q ← {(V_i,V_{cv}) in arcs(G),i<cv}   % arcs to labelled variables.
  consistent ← true
  while consistent & Q non empty **do**
    select and delete any arc (V_k,V_m) from Q
    consistent ← not REVISE(V_k,V_m)
  end while
  return consistent
end AC-BT

When a value is deleted, the domain is empty

Backjumping & comp. uses information about the violated constraints.

Forward checking

It is better to prevent failures than to detect them only!

Consistency techniques can **remove incompatible values** for future (=not yet labelled) variables.

Forward checking ensures consistency between the currently labelled variable and the variables connected to it via constraints.

Forward consistency checks

procedure AC-FC(G,cv)
  Q ← {(V_i,V_{cv}) in arcs(G),i>cv}   % arcs to future variables
  consistent ← true
  while consistent & Q non empty **do**
    select and delete any arc (V_k,V_m) from Q
    if REVISE(V_k,V_m) **then**
      consistent ← not empty D_k
    end if
  end while
  return consistent
end AC-FC

Empty domain implies inconsistency

Partial look ahead

We can extend the consistency checks to more future variables!

The value assigned to the current variable can be propagated to all future variables.

Partial lookahead consistency checks

```
procedure DAC-LA(G,cv)
    for i=cv+1 to n do
        for each arc (Vi,Vj) in arcs(G) such that i>j & j≥cv do
            if REVISE(Vi,Vj) then
                if empty Di then return fail
        end for
    end for
    return true
end DAC-LA
```

Notes:

- In fact DAC is maintained (in the order reverse to the labelling order).
  - **Partial Look Ahead** or **DAC - Look Ahead**
- It is not necessary to check consistency of arcs between the future variables and the past variables (different from the current variable)!

Full look ahead

Knowing more about far future is an advantage!

Instead of DAC we can **use a full AC** (e.g. AC-3).

Full look ahead consistency checks

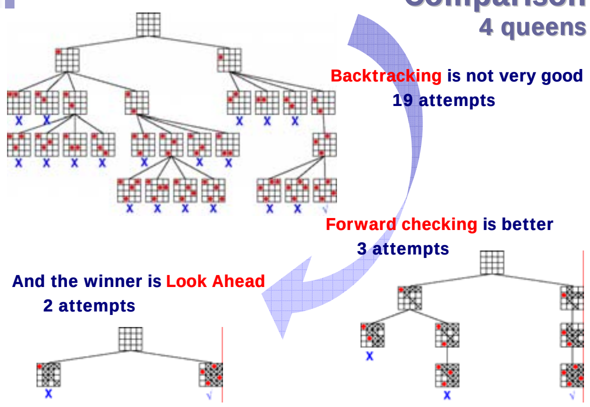
```
procedure AC3-LA(G,cv)
    Q ← {(Vi,Vcv) in arcs(G),i>cv}          % start with arcs going to cv
    consistent ← true
    while consistent & Q non empty do
        select and delete any arc (Vk,Vm) from Q
        if REVISE(Vk,Vm) then
            Q ← Q ∪ {(Vi,Vk) | (Vi,Vk) in arcs(G),i≠k,i≠m,i>cv}
            consistent ← not empty Dk
        end if
    end while
    return consistent
end AC3-LA
```

Notes:

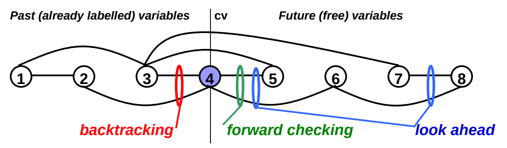
- The arcs going to the current variable are checked exactly once.
- The arcs to past variables are not checked at all.
- It is possible to use other than AC-3 algorithms (e.g. AC-4)

Comparison

4 queens



Constraint propagation at glance



- Propagating through more constraints remove more inconsistencies (BT < FC < PLA < LA), of course it increases complexity of the step.
- Forward Checking does no increase complexity of backtracking, the constraint is just checked earlier in FC (BT tests it later).
- When using AC-4 in LA, the initialisation is done just once.
- Consistency can be ensured before starting search
  - **Algorithm MAC (Maintaining Arc Consistency)**
    - AC is checked before search and after each assignment
- It is possible to use stronger consistency techniques (e.g. use them once before starting search).

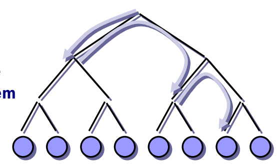
Consistency and Search

Consistency techniques are (usually) incomplete.

↳ We need a search algorithm to resolve the rest!

Labeling

- **depth-first search**
  - **assign a value to the variable**
  - **propagate = make the problem locally consistent**
  - **backtrack upon failure**



- **X in 1..3** ≈ **X=1 ∨ X=2 ∨ X=3 (enumeration)**

In general, search algorithm resolves remaining disjunctions!

- **X=1 ∨ X≠1** **(step labeling)**
- **X<3 ∨ X≥3** **(bisection)**
- **X<Y ∨ X≥Y** **(variable ordering)**

Variable ordering

Variable ordering in labelling influence significantly efficiency of solvers (e.g. in a tree-structured CSP).

What variable ordering should be chosen in general?

FAIL-FIRST principle

„select the variable whose instantiation will lead to a failure“

it is better to tackle failures earlier, they can be become even harder

- **prefer the variables with smaller domain** (dynamic order)
  - a smaller number of choices ~ lower probability of success
  - the dynamic order is appropriate only when new information appears during solving (e.g., in look ahead algorithms)

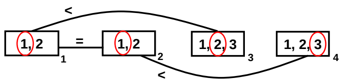
„solve the hard cases first, they may become even harder later“

- **prefer the most constrained variables**
  - it is more complicated to label such variables (it is possible to assume complexity of satisfaction of the constraints)
  - this heuristic is used when there is an equal size of the domains
- **prefer the variables with more constraints to past variables**
  - a static heuristic that is useful for look-back techniques

Backtrack-free search

Definition:

CSP is solved using backtrack-free search if for some order of variables we can find a value for each variable compatible with the values of already assigned variables.



How to find out a sufficient consistency level for a given graph?

Some observations:

- variable must be compatible with all the "former" variables i.e., across the „backward" edges
- for k „backward" edges we need (k+1)-consistency
- let m be the maximum of backward edges for all the vertices, then strong (m+1)-consistency is enough
- the number of backward edges is different for different variable order
- of course, the order minimising m is looked for

Value ordering

Order of values in labelling influence significantly efficiency (if we choose the right value each time, no backtrack is necessary).

What value ordering for the variable should be chosen in general?

SUCCEED FIRST principle

„prefer the values belonging to the solution"

- if no value is part of the solution then we have to check all values
- if there is a value from the solution then it is better to find it soon

Note: SUCCEED FIRST does not go against FIRST-FAIL !

- **prefer the values with more suppors**
  - this information can be found in AC-4
- **prefer the value leading to less domain reduction**
  - this information can be computed using singleton consistency
- **prefer the value simplifying the problem**
  - solve approximation of the problem (e.g. a tree)

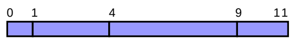
Generic heuristics are usually too complex for computation.

It is better to use problem-driven heuristics that propose the value!

Example

Golomb ruler

- A **ruler with M marks** such that **distances** between any two marks are **different**.
- The **shortest ruler** is the optimal ruler.



- **Hard** for M≥16, no exact algorithm for M ≥ 24!
- Applied in **radioastronomy**.

Solomon W. Golomb
Professor
University of Southern California
http://csi.usc.edu/faculty/golomb.html

Golomb ruler
CSP model

A base model:

Variables X_1, ..., X_M with the domain 0..M*M

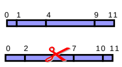
$X_1 = 0$ — *ruler start*

$X_1 < X_2 < \ldots < X_M$ — *no permutations of variables*

$\forall i<j \; D_{i,j} = X_j - X_i$ — *difference variables*

all_different($\{D_{1,2}, D_{1,3}, \ldots D_{1,M}, D_{2,3}, \ldots D_{M,M-1}\}$)

Model extensions:



$D_{1,2} < D_{M-1,M}$ — *symmetry breaking*

better bounds (**implied constraints**) for $D_{i,j}$

$D_{i,j} = D_{i,i+1} + D_{i+1,i+2} + \ldots + D_{j-1,j}$

so $D_{i,j} \geq \Sigma_{j-i} = (j-i)*(j-i+1)/2$ — *lower bound*

$X_M = X_M - X_1 = D_{1,M} = D_{1,2} + D_{2,3} + \ldots D_{i-1,i} + D_{i,j} + D_{j,j+1} + \ldots + D_{M-1,M}$

$D_{i,j} = X_M - (D_{1,2} + \ldots D_{i-1,i} + D_{j,j+1} + \ldots + D_{M-1,M})$

so $D_{i,j} \leq X_M - (M-1-j+i)*(M-j+i)/2$ — *upper bound*

Golomb ruler
some results

- **What is the effect of different constraint models?**

size	base model	base model + symmetry	base model + symmetry + implied constraints
7	220	80	30
8	1 462	611	190
9	13 690	5 438	1 001
10	120 363	49 971	7 011
11	2 480 216	985 237	170 495

time in milliseconds on Mobile Pentium 4-M 1.70 GHz, 768 MB RAM

- **What is the effect of different search strategies?**

size	fail first			leftmost first		
	enum	*step*	*bisect*	*enum*	*step*	*bisect*
7	40	60	40	30	30	30
8	390	370	350	220	190	200
9	2 664	2 384	2 113	1 182	1 001	921
10	20 870	17 545	14 982	8 782	7 011	6 430
11	1 004 515	906 323	779 851	209 251	170 495	159 559

time in milliseconds on Mobile Pentium 4-M 1.70 GHz, 768 MB RAM

Conclusions

Constraint solvers

- It is not necessary to program all the presented techniques from scratch!
- Use existing constraint solvers (packages)!
  - provide **implementation of data structures** for modeling variables' domains and constraints
  - provide a basic **consistency framework** (AC-3)
  - provide **filtering algorithms** for many constraints (including global constraints)
  - provide basic **search strategies**
  - usually **extendible** (new filtering algorithms, new search strategies)

Some systems with **constraint satisfaction packages**:

- **Prolog**: CHIP, ECLiPSe, SICStus Prolog, Prolog IV, GNU Prolog, IF/Prolog
- **C/C++**: CHIP++, ILOG Solver
- **Java**: JCK, JCL, Koalog
- Mozart

86

Resources

- **Books**
  - P. Van Hentenryck: **Constraint Satisfaction in Logic Programming**, MIT Press, 1989
  - E. Tsang: **Foundations of Constraint Satisfaction**, Academic Press, 1993
  - K. Marriott, P.J. Stuckey: **Programming with Constraints: An Introduction**, MIT Press, 1998
  - T. Frühwirth, S. Abdennadher: **Essentials of Constraint Programming**, Springer Verlag, 2003
  - R. Dechter: **Constraint Processing**, Morgan Kaufmann, 2003
- **Journal**
  - Constraints, An International Journal. Kluwer Academic Publishers (Springer)
- **On-line materials**
  - **On-line Guide to Constraint Programming** (tutorial) http://kti.mff.cuni.cz/~bartak/constraints/
  - **Constraints Archive** (archive and links) http://4c.ucc.ie/web/archive/index.jsp
  - **Constraint Programming online** (community web) http://www.cp-online.org/

87

Summary

Constraints

- arbitrary relations over the problem variables
- express partial local information in a declarative way

Basic constraint satisfaction framework:

- **local consistency** connecting filtering algorithms for individual constraints
- **depth-first search** resolves remaining disjunctions
- **local search** can also be used

Problem solving using constraints:

- **declarative modeling** of problems as a CSP
- **dedicated algorithms** can be encoded in constraints
- special **search strategies**

It is easy to state combinatorial problems in terms of a CSP
... but it is more complicated to design solvable models.

We still did not reach **the Holy Grail** of computer programming (the user states the problem, the computer solves it) but CP is close.

88